Spring 2014 **Math 601** Name: ______________________

Quiz 9

Question 1. (10 pts)

Evaluate the integral

$$\int_C \frac{z^2 + \sin z}{z^2 - 7z + 6} dz$$

(a) when C is the circle $|z - 6| = 2$, that is, the circle centered at 6 with radius 2.

> **Solution:** Notice that
>
> $$\frac{z^2 + \sin z}{z^2 - 7z + 6} = \frac{z^2 + \sin z}{(z-1)(z-6)}$$
>
> So $z = 6$ is inside the circle $|z - 6| = 1$ and $z = 1$ is outside the circle $|z - 6| = 1$. By Cauchy's formula, we have
>
> $$\int_C \frac{z^2 + \sin z}{z^2 - 7z + 6} dz = \int_C \frac{\frac{z^2 + \sin z}{(z-1)}}{z - 6} dz = 2\pi i \frac{36 + \sin 6}{5}.$$

(b) when C is the circle $|z| = 3$, that is, the circle centered at 0 with radius 3.

> **Solution:** This time, $z = 1$ is inside the circle $|z| = 3$ and $z = 6$ is outside the circle $|z| = 3$. By Cauchy's formula, we have
>
> $$\int_C \frac{z^2 + \sin z}{z^2 - 7z + 6} dz = \int_C \frac{\frac{z^2 + \sin z}{(z-6)}}{z - 1} dz = 2\pi i \frac{1 + \sin 1}{-5}.$$

Question 2. (10 pts)

Show that

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2-1}dz = \frac{e^t - e^{-t}}{2}$$

where C is the the circle $|z| = 2$.

Solution:

$$\frac{e^{zt}}{z^2-1} = \frac{e^{zt}}{(z+1)(z-1)}$$

Both 1 and -1 lie inside the circle $|z| = 2$. Consider the partial fractions

$$\frac{1}{(z+1)(z-1)} = \frac{A}{z-1} - \frac{B}{z+1} = \frac{A(z+1) - B(z-1)}{(z+1)(z-1)}$$

So $A = B = \frac{1}{2}$. Now apply Cauchy's formula

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2-1}dz \\
&= \frac{1}{2\pi i}\int_C \frac{e^{zt}}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz \\
&= \frac{e^t}{2} - \frac{e^{-t}}{2}
\end{aligned}$$